Eur. Phys. J. B **12**, 285–297 (1999)

THE EUROPEAN PHYSICAL JOURNAL B
EDP Sciences

The nonlinear Fokker-Planck equation with state-dependent diffusion – a nonextensive maximum entropy approach

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Received 26 February 1999

Abstract. Nonlinear Fokker-Planck equations (*e.g.*, the diffusion equation for porous medium) are important candidates for describing anomalous diffusion in a variety of systems. In this paper we introduce such nonlinear Fokker-Planck equations with general state-dependent diffusion, thus significantly generalizing the case of constant diffusion which has been discussed previously. An approximate maximum entropy (MaxEnt) approach based on the Tsallis nonextensive entropy is developed for the study of these equations. The MaxEnt solutions are shown to preserve the functional relation between the time derivative of the entropy and the time dependent solution. In some particular important cases of diffusion with power-law multiplicative noise, our MaxEnt scheme provides *exact* time dependent solutions. We also prove that the stationary solutions of the nonlinear Fokker-Planck equation with diffusion of the (generalized) Stratonovich type exhibit the Tsallis MaxEnt form.

PACS. 66.10.Cb Diffusion and thermal diffusion – 05.20.-y Classical statistical mechanics – 05.60.-k Transport processes

1 Introduction

The extensivity of entropy is one of the basic assumptions of standard thermodynamics and statistical mechanics. However, some of the currently most active areas of research in statistical physics deal with systems that stubbornly refuse to follow the strictures of the extensivity paradigm [1]. A case example is provided by self-gravitating systems [2]. Astrophysicists have tried for decades to develop a thermostatistical description of self-gravitating systems along the lines of standard statistical mechanics [3,4]. The failure of those attempts was due to the nonextensivity effects associated with the long range of the gravitational interaction [4].

A new entropy functional introduced a few years ago by Tsallis [5] is nowadays regarded as the possible basis for a generalized thermostatistics [6] appropriate to deal with nonextensive settings [7]. This entropy has the form

$$S_q = \frac{1}{q-1}\left(1 - \int f(\mathbf{x})^q \, \mathrm{d}\mathbf{x}\right), \tag{1}$$

where $\mathbf{x} \in R^N$ is a dimensionless state-variable, f corresponds to the probability distribution and the entropic index q is any real number. This entropy recovers the standard Boltzmann-Gibbs entropy $S = -\int f(\mathbf{x}) \ln f(\mathbf{x}) \mathrm{d}\mathbf{x}$ in the limit $q \to 1$. The measure S_q is *nonextensive* such that $S_q(A+B) = S_q(A) + S_q(B) + (1-q)S_q(A)S_q(B)$, where A and B are two systems independent in the sense that $f(\mathbf{x}, \mathbf{x}')_{A+B} = f(\mathbf{x})_A f(\mathbf{x}')_B$. It is clear that q can be seen as measuring the degree of nonextensivity. Many relevant mathematical properties of the standard thermostatistics still hold true within Tsallis' formalism, or admit natural generalizations [8–12]. Tsallis' proposal was shown to be consistent both with Jaynes' Information Theory formulation of statistical mechanics [13], and with the dynamical thermostatting approach to statistical ensembles [14].

The recent application of Tsallis' theory to an increasing number of physical problems is beginning to provide a picture of the kind of scenarios where the new formalism is useful. Self-gravitating systems constituted the first physical problem discussed within the nonextensive thermostatistics [15]. That early application, in turn, inspired Boghosian's treatment of the two-dimensional pure electron plasma, yielding the first experimental confirmation of Tsallis' theory [16]. A possible solution of the solar neutrino puzzle based on Tsallis thermostatistics has been advanced [17]. Some cosmological implications of Tsallis' proposal have also been worked out [18]. Tsallis statistics has been succesfuly applied to the peculiar velocity

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distribution of galaxy clusters [19] as well as to the phase shift analysis for the pion-nucleus scattering [20]. The behaviour of dissipative low dimensional chaotic systems [21,22], Hamiltonian chaotic systems with long range interactions [23], as well as self organized critical systems [24], have been discussed in connection with the new approach. The Tsallis entropy has also been advanced as the basis of a thermostatistical foundation of Lévy flights and distributions [25]. The development of powerful optimization algorithms based on Tsallis thermostatistics [26] constitutes another remarkable achievement of the new formalism.

Another interesting feature is that exact time dependent solutions for a family of nonlinear Fokker-Planck equations can be obtained by the maximization of Tsallis' entropy under appropriate constraints [27]. One of us has recently shown that Tsallis maximum entropy (MaxEnt) distributions also provide stationary solutions for *linear* Fokker-Planck equations characterized by appropriate drift forces [28] (see also [29]). However, as recently pointed out by Jespersen and his collaborators [30], Tsallis MaxEnt distributions arise in a more natural way within the context of nonlinear Fokker-Planck equations. The main difference between the nonlinear Fokker-Planck equations and the standard linear ones is that the diffusion term depends on a power of the probability density. These nonlinear Fokker-Planck equations admit important physical applications such as the percolation of gases through porous media [31], thin liquid films spreading under gravity [32], surface growth [33], and some self-organizing phenomena [34]. In particular, they are used to describe systems showing anomalous diffusion of the correlated type [35,36]. Anomalous diffusion is characterized by the fact that the mean square displacement of the relevant state variable – written here for the one-dimensional variable x – scales as $\langle x^2\rangle(t) \propto t^\gamma$, where the diffusion exponent $\gamma = 1$ yields normal diffusion, $\gamma < 1$ corresponds to subdiffusion and $\gamma > 1$ corresponds to superdiffusion.

The solutions of the nonlinear Fokker-Planck equation studied in [27] correspond to the one-dimensional case of a constant diffusion coefficient along with a linear homogeneous drift force. These analytical solutions maximize the Tsallis entropy under the constraints imposed by normalization and the mean value of x and x^2. They are natural generalizations (in the sense of Tsallis' formalism) of the well known Gaussian solutions of a (linear) Ornstein-Uhlenbeck process [37–41]. These results where later generalized to the the case of a linear non-homogeneous drift [42]. The Tsallis MaxEnt solutions were recently used to study the problem of aging [43] (see, however, [44]). A phenomenological microscopic approach to the nonlinear Fokker-Planck equation, based on an appropriate generalization of Ito-Langevin dynamics, was recently developed by one of us [44].

The existence of time dependent solutions of a Tsallis maximum entropy form suggests that there is an intimate relationship between the nonlinear Fokker-Planck equation and Tsallis thermostatistics. The aim of the present effort is to pursue a further exploration of that connection. The paper is organized as follows. In Section 2 we provide a brief review of Jaynes approach to time dependent problems. Section 3 deals with some properties of the nonlinear Fokker-Planck equation, in particular with respect to state-dependent diffusion. In Section 4 we consider the MaxEnt approach to the nonlinear Fokker-Planck equation based on the Tsallis entropy. In Section 5 we show some particular cases where the nonlinear Fokker-Planck equation admits exact solutions of the Tsallis maximum entropy form. Anomalous diffusion properties for these cases are also discussed. Finally, some conclusions are given in Section 6.

2 Maximum entropy approach to evolution equations

The main idea of Jaynes maximum entropy approach to evolution equations is to focus on the behaviour of the mean values of a rather small number of relevant dynamical quantities, instead of trying to follow the temporal evolution of the system in all its full detail. The probability distribution function (or statistical operator, in the case of quantum mechanics) adopted to describe the system is then the one that maximizes the Shannon entropy ($S = -\int f(\mathbf{x}) \ln f(\mathbf{x}) \mathrm{d}\mathbf{x}$ or $S = -\mathrm{Tr}(\rho \ln \rho)$) under the constraints imposed by normalization and the mean values of the relevant quantities. This approach was introduced by Jaynes in order to provide a new formulation of Statistical Mechanics based on Information Theory [45–48]. The evolution equations considered by Jaynes were the von Neuman equation and the Liouville equation for Hamiltonian systems. Within that context, the constants of motion of the system constitute the most natural set of relevant mean values. That choice allows one to obtain all the statistical ensembles that appear in equilibrium statistical mechanics. However, one of the most interesting possibilities allowed by Jaynes' formulation is to include non-conserved quantities within the set of relevant observables, in order to describe non-equilibrium situations. For instance, let us consider a quantum system with Hamiltonian $\hat{H}$ described by the density operator $\hat{\rho}$. The evolution of the density operator is given by the von Neuman equation

$$\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = -\mathrm{i}[\hat{H}, \hat{\rho}], \tag{2}$$

where Planck's constant was set equal to 1. The Ehrenfest theorem provides the time derivative of the expectation value of an observable $\hat{O}_i$

$$\frac{\mathrm{d}\langle\hat{O}_i\rangle}{\mathrm{d}t} = \mathrm{i}\langle[\hat{H}, \hat{O}_i]\rangle. \tag{3}$$

Now, suppose we have a set of M observables $\hat{O}_i$, $i = 1, \ldots, M$ that close a semialgebra under commutation with the Hamiltonian operator

$$[\hat{H}, \hat{O}_i] = \sum_{j=1}^{M} g_{ij} \hat{O}_j, \quad i = 1, \ldots, M, \tag{4}$$

where g_{ij} are the concomitant structure constants. In that case, the mean values of the M observables evolve according to a closed set of linear ordinary differential equations,

$$\frac{\mathrm{d}\langle \hat{O}_i \rangle}{\mathrm{d}t} = \mathrm{i} \sum_{j=1}^{M} g_{ij} \langle \hat{O}_j \rangle, \quad i = 1, \ldots, M. \tag{5}$$

In this case, the M observables $\langle \hat{O}_i \rangle$ are as good as integrals of motions in order to implement Jaynes approach [49–52]. Solving the linear differential equations (5), the expectation values $\langle \hat{O}_i \rangle(t)$ at any time t can be computed from their values at an initial time t_0. Hence, the concomitant MaxEnt statistical operator $\hat{\rho}_{\mathrm{ME}}$ can be obtained at any time t, as happens in the equilibrium situation. Indeed, it can be shown that this MaxEnt density operator is an exact solution to the von Neuman equation.

Of course, the closure condition (4) does not always hold. However, even when the relevant observables $\hat{O}_i$ do not close a semialgebra with the Hamiltonian, we can still close the set of equations (3) in a nonlinear and approximate way by recourse to the maximum entropy approach. We can evaluate the expectation values appearing on the right hand sides of equations (3) using a maximum entropy density matrix $\hat{\rho}_{\mathrm{Me}}$ determined, at each time t, by the constraints imposed by the M instantaneous expectation values $\langle \hat{O}_i \rangle$. For an example of this scheme, let us consider a quantum many body system. If the set of relevant observables $\hat{O}_i$ consists only of one-particle operators, then the maximum entropy approximate closure approach yields the well known time dependent Hartree-Fock approximation.

Within the classical domain, these MaxEnt ideas have been applied to a variety of evolution equations. A maximum entropy scheme has been numerically implemented for the cosmic ray transport equation [53]. The MaxEnt approach has been applied to the Liouville equation associated with dynamical systems showing divergenceless phase space flows [54], to the (linear) Fokker-Planck equation [55], and to a more general family of evolution equations with the form of linear continuity equations [56]. The inverse problem of reconstructing the underlying microscopic dynamics from time-series using maximum entropy ideas has also been addressed [57].

All the above referred applications of Jaynes' approach are based on the analysis of the evolution of the information content associated with the set of relevant mean values. This information evolution is given by the behaviour of those mean values, that is determined in turn by the evolution equation under consideration.

3 The nonlinear Fokker-Planck equation

3.1 The operators L_D and L_R

Now let us focus on the nonlinear, generalized Fokker-Planck equation, which is given by

$$\frac{\partial f}{\partial t} = L_{\mathrm{R}} f + L_{\mathrm{D}} \left(f^{\alpha} \right). \tag{6}$$

Here $f(\mathbf{x}, t)$ is the normalized distribution function, L_{R} and L_{D} are linear differential operators. We shall often use the notation $\alpha = 2 - q$ which connects the nonlinearity in the diffusion equation with the entropic index q parametrizing the Tsallis entropy. This result is known from earlier work [27,42] where it was shown that a particular solution of the nonlinear Fokker-Planck equation characterized by α maximizes the Tsallis entropy of index $q = 2 - \alpha$. The drift term

$$L_{\mathrm{R}} f = -\sum_{i=1}^{N} \frac{\partial (K_i f)}{\partial x_i} \tag{7}$$

is due to the deterministic forces associated to the drift vector $\mathbf{K}(\mathbf{x}) \in R^N$ of components K_i, while the nonlinear diffusion term

$$L_{\mathrm{D}} \left(f^{2-q} \right) = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left(D_{ij}(\mathbf{x}) \frac{\partial f^{2-q}}{\partial x_j} \right) \tag{8}$$

describes the effect of stochastic forces characterized by the diffusion tensor $\mathbf{D}(\mathbf{x}) \in R^{N \times N}$, of components D_{ij}. Note that the diffusion coefficients may depend on the state variable $\mathbf{x}$, thus generalizing the previously studied nonlinear Fokker-Planck equation with *constant* diffusion. In the limit case $q \to 1$, we recover the N-dimensional *linear* Fokker-Planck equation,

$$\frac{\partial f}{\partial t} = L_{\mathrm{FP}} f, \tag{9}$$

which can be written in terms of just one single linear differential operator $L_{\mathrm{FP}} = L_{\mathrm{R}} + L_{\mathrm{D}}$.

It will be useful for our later discussions to introduce the adjoint operators $L_{\mathrm{R}}^{\dagger}$ and $L_{\mathrm{D}}^{\dagger}$, defined by

$$\int f_(L_{\mathrm{R}} f_2) \mathrm{d}\mathbf{x} = \int (L_{\mathrm{R}}^{\dagger} f_1) f_2 \mathrm{d}\mathbf{x}, \tag{10}$$

and

$$\int f_1 (L_{\mathrm{D}} f_2) \mathrm{d}\mathbf{x} = \int (L_{\mathrm{D}}^{\dagger} f_1) f_2 \mathrm{d}\mathbf{x}, \tag{11}$$

for any two probability distributions f_1 and f_2. These adjoint operators are

$$L_{\mathrm{R}}^{\dagger} = \sum_{i=1}^{N} K_i \frac{\partial}{\partial x_i} \tag{12}$$

and,

$$L_{\rm D}^{\dagger} = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_j} D_{ij}(\mathbf{x}) \frac{\partial}{\partial x_i} \cdot \qquad (13)$$

Defined in a similar way, the adjoint operator $L_{\rm FP}^{\dagger} = L_{\rm R}^{\dagger} + L_{\rm D}^{\dagger}$ plays an important role in the study of the *linear* Fokker-Planck equation [39].

3.2 Generalized Stratonovich and Ito forms

Before we go on, let us first remark on other possible forms of a nonlinear Fokker-Planck equation with $\mathbf{x}$-dependent diffusion tensor $D_{ij}(\mathbf{x})$. To this end, we revisit the standard linear Fokker-Planck equation. For the sake of simplicity we will focus on the one-dimensional situation, but our considerations on this point are also valid in the N-dimensional case. For the linear Fokker-Planck equation, there are two possible forms for state-dependent diffusion, namely:

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x}(K(x)f) + \frac{\partial}{\partial x}\left(D(x)\frac{\partial f}{\partial x}\right), \qquad (14)$$

which is the Stratonovich form (here K and D stand, respectively, for the single components of the drift vector and the diffusion constant) and

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x}(K(x)f) + \frac{\partial^2}{\partial x^2}(D(x)f), \qquad (15)$$

which is the Ito-form. Even in the linear case there has been some discussion on which of these two forms should be used (see for example [58]). Nevertheless, it should be pointed out that these two forms can formally be mapped into each other

$$\begin{aligned} \frac{\partial f}{\partial t} &= -\frac{\partial}{\partial x}(Kf) + \frac{\partial^2}{\partial x^2}(D(x)f) & (16)\\ &= -\frac{\partial}{\partial x}(Kf) + \frac{\partial}{\partial x}\left(\frac{\partial D}{\partial x}f\right) + \frac{\partial}{\partial x}\left(D(x)\frac{\partial f}{\partial x}\right) & (17)\\ &= -\frac{\partial}{\partial x}(\tilde{K}f) + \frac{\partial}{\partial x}\left(D(x)\frac{\partial f}{\partial x}\right), & (18) \end{aligned}$$

where $\tilde{K} = K - \frac{\partial D}{\partial x}$ now includes a noise-induced drift term.

However, in the nonlinear case, the corresponding Ito and Stratonovich forms are no longer equivalent. The Stratonovich form corresponds to that of equation (6), which in the one-dimensional case reads

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x}(K(x)f) + \frac{\partial}{\partial x}\left(D(x)\frac{\partial f^{\alpha}}{\partial x}\right), \qquad (19)$$

whereas the Ito form would be of type

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x}(K(x)f) + \frac{\partial^2}{\partial x^2}\left(D(x)f^{\alpha}\right). \qquad (20)$$

It is simple to see as follows that the Ito form can *not* be mapped onto the Stratonovich form, *nor* does it have any trivial solutions.

As we shall discuss again in Section 4, the stationary solutions of the Stratonovich equation (19) are of the form

$$f(x) = \varGamma[1-\beta(1-q)V_{\rm eff}(x)]^{1/(1-q)}, \qquad (21)$$

with the effective potential

$$V_{\rm eff} = -\int \frac{K(x)}{D(x)}{\rm d}x. \qquad (22)$$

Here, β and $\varGamma$ are appropriately defined constants. These solutions are such that they maximize the nonextensive generalized Tsallis entropy (1) under simple constraints, and are therefore referred to as Tsallis distributions.

The stationary solutions of the Ito form (20) must on the other hand satisfy

$$-K(x)f + \frac{\partial D(x)}{\partial x}f^{\alpha} = -D(x)\frac{\partial f^{\alpha}}{\partial x}, \qquad (23)$$

where we have assumed appropriate boundary conditions on f. This equation does not allow for separation of the variables f and x in any trivial form. Furthermore, whichever (complicated) solutions one may find are definitely not of the Tsallis form. For these reasons we prefer to work with the nonlinear state-dependent diffusion equation of (6). Another argument in favour of the Stratonovich form is that even in the linear case it is considered more physically relevant than the Ito form because it gives a more realistic treatment of noise [39,40].

The nonlinear generalization of the Stratonovich form provided by equation (6) shows, in the case of pure diffusion (*i.e.*, vanishing drift), another interesting property related to the time derivative ${\rm d}S_q[f]/{\rm d}t$ of the Tsallis entropy S_q of equation (1). It is easy to prove that the time derivative of S_q, evaluated on a particular solution $f(x,t)$ of equation (6), is given by

$$\frac{{\rm d}S_q}{{\rm d}t} = q(2-q)\int \frac{D(x)}{f(x,t)}\left(\frac{\partial f}{\partial x}\right)^2 {\rm d}x, \qquad (24)$$

where we have again used $\alpha = 2-q$. Up to the constant factor $q(2-q)$, this expression coincides with the one that holds in the usual case of linear diffusion. In the particular and important case of constant diffusion we have

$$\frac{{\rm d}S_q}{{\rm d}t} = q(2-q)DI[f], \qquad (25)$$

where $I[f]$ stands for the Fisher Information associated with the distribution f,

$$I[f] = \int \frac{1}{f}\left(\frac{\partial f}{\partial x}\right)^2 {\rm d}x. \qquad (26)$$

Hence, we see that the time derivative of Tsallis entropy S_q behaves essentially in the same way as the standard logarithmic measure does in the case of linear diffusion.

4 Generalized maximum entropy approach

4.1 q-maximum entropy approximation

As mentioned in Section 2, the main idea of the MaxEnt approach to time dependent probability distribution functions is to focus on the behaviour of a relatively small number of relevant mean values, instead of trying to follow in detail the evolution of the complete distribution $f(\mathbf{x},t)$. Here we shall reformulate these ideas within the Tsallis generalized framework. Let us consider a set of M q-generalized mean values

$$\langle A_i\rangle_q = \int f^q A_i(\mathbf{x})\mathrm{d}\mathbf{x} \quad (i=1,.....,M). \tag{27}$$

Before proceeding we mention that the formalism which we now develop can alternatively be obtained using the normalized mean values $\langle\langle A_i\rangle\rangle_q \equiv \langle A_i\rangle_q/\int f^q\mathrm{d}\mathbf{x}$ [59] as well as by using the standard mean values $\langle A_i\rangle = \int A_i f\mathrm{d}\mathbf{x}$. In the latter case all results will stay the same except for a sign change ($q-1$ must be replaced by $1-q$). It is important to stress here that, among these three types of mean values, the normalized q-mean values $\langle\langle A_i\rangle\rangle_q$ are the ones best suited in order to develop a well-behaved thermostatistical formalism [59]. However, due to the fact that some important features are shared by the three kinds of mean values (namely, the power-law form of Tsallis MaxEnt distributions and the Legendre transform structure of the concomitant thermostatistical formalism) [59], the results of the present paper can be formulated in terms of any of those three types of mean values. For the sake of mathematical convenience we choose to work with the mean values given by (27). Within the present application of Tsallis theory, the formalism based upon the normalized q-mean values leads to the same results as the ones we obtain here, but with an unnecessary complication of the mathematical calculations involved.

If the evolution is governed by the nonlinear Fokker-Planck equation, then the time derivatives of these M moments of $f(\mathbf{x},t)$ are given by

$$\frac{\mathrm{d}\langle A_i\rangle_q}{\mathrm{d}t} = q\int\left\{L_{\mathrm{R}}f + L_{\mathrm{D}}\left(f^{2-q}\right)\right\} f^{q-1}A_i\mathrm{d}\mathbf{x} \qquad (i=1,\ldots,M). \tag{28}$$

Unfortunately, these equations do not, in general, constitute a closed system of ordinary differential equations of motion for the mean values $\langle A_i\rangle_q$. The integrals appearing on the right hand sides of equations (28) are not, in general, equal to linear combinations of the original mean values $\langle A_i\rangle_q$. Here enters the maximum entropy principle. We can "close" the set (28), in an approximate way, by evaluating the right hand sides using, at each instant of time, the MaxEnt distribution $f_{\mathrm{ME}}(\mathbf{x},t)$ that maximizes Tsallis' entropy (1) under the constraints imposed by normalization and the M instantaneous values of $\langle A_i\rangle_q$. The concomitant variational problem has a well-known analytical solution

$$f_{\mathrm{ME}}(\mathbf{x},t) = Z_q^{-1}(t)\left[1-(1-q)\sum_{i=1}^{M}\lambda_i(t)A_i(\mathbf{x})\right]^{\frac{1}{1-q}}, \tag{29}$$

where the (λ_i, $i=1,\ldots,M$) are appropriate Lagrange multipliers that guarantee compliance with the given constraints, and Z_q is the partition function given by

$$Z_q = \int\left[1-(1-q)\sum_{i=1}^{M}\lambda_i A_i(\mathbf{x})\right]^{\frac{1}{1-q}}\mathrm{d}\mathbf{x}. \tag{30}$$

The generalized entropy S_q, the partition function Z_q, the generalized mean values, and the concomitant Lagrange multipliers, are related by Jaynes' thermodynamical relations,

$$\frac{\partial S_q}{\partial\langle A_i\rangle_q} = \lambda_i, \tag{31}$$

and

$$\frac{\partial\lambda_J}{\partial\lambda_i} = -\langle A_i\rangle_q, \tag{32}$$

where the (q-generalized) Jaynes' parameter λ_J is given by

$$\lambda_J = \ln_q Z_q = \frac{Z_q^{1-q}-1}{1-q}\cdot \tag{33}$$

4.2 Evolution of the generalized entropy S_q

The time derivative of the entropy constitutes one of the most important qualitative features characterizing the behaviour of probability density functions associated with irreversible phenomena. When that derivative has a definite sign, an H-theorem holds, and a useful mathematical realization of "the arrow of time" becomes available. Given an approximate scheme for solving the evolution equations describing the system under study, it is crucial to know how close the behaviour of the entropy evaluated on the approximate solutions follows the evolution of the entropy of the exact solutions. Here we shall compare the behaviour of the Tsallis entropy S_q corresponding to exact solutions of the nonlinear Fokker-Planck equation, with the behaviour associated to our MaxEnt solutions.

If $f(\mathbf{x},t)$ is an exact solution of (1), we have

$$\frac{\mathrm{d}S_q}{\mathrm{d}t} = \frac{q}{1-q}\int f^{q-1}\left\{L_{\mathrm{R}}f + L_{\mathrm{D}}\left(f^{2-q}\right)\right\}\mathrm{d}\mathbf{x}, \tag{34}$$

which, after introducing the functional

$$D_q[f] = \frac{q}{1-q}\int f^{q-1}\left\{L_{\mathrm{R}}f + L_{\mathrm{D}}\left(f^{2-q}\right)\right\}\mathrm{d}\mathbf{x}, \tag{35}$$

can be cast under the guise

$$\frac{\mathrm{d}S_q}{\mathrm{d}t} = D_q[f]. \tag{36}$$

This last expression is preserved by the maximum entropy approach, as can be seen as follows. We have

$$\frac{\mathrm{d}S_q\left[f_{\mathrm{ME}}\right]}{\mathrm{d}t} = \sum_{i=1}^{M} \frac{\partial S_q}{\partial \langle A_i \rangle_q} \frac{\mathrm{d}\langle A_i \rangle_q}{\mathrm{d}t} = \sum_{i=1}^{M} \lambda_i \frac{\mathrm{d}\langle A_i \rangle_q}{\mathrm{d}t} \quad (37)$$

which yields

$$\frac{\mathrm{d}S_q\left[f_{\mathrm{ME}}\right]}{\mathrm{d}t} = q \int \left\{ L_{\mathrm{R}} f_{\mathrm{ME}} + L_{\mathrm{D}} \left(f_{\mathrm{ME}}^{2-q} \right) \right\} f_{\mathrm{ME}}^{q-1} \left[\sum_{i=1}^{M} \lambda_i \; A_i \right] \mathrm{d}\mathbf{x}. \quad (38)$$

But

$$\sum_{i=1}^{M} \lambda_i \; A_i = \frac{1}{1-q} \; \left[1 - Z_q^{1-q} f_{\mathrm{ME}}^{1-q} \right] \quad (39)$$

so that

$$\frac{\mathrm{d}S_q\left[f_{\mathrm{ME}}\right]}{\mathrm{d}t} = \frac{q}{1-q} \int \left\{ L_{\mathrm{R}} f_{\mathrm{ME}} + L_{\mathrm{D}} \left(f_{\mathrm{ME}}^{2-q} \right) \right\} \left\{ f_{\mathrm{ME}}^{q-1} - Z_q^{1-q} \right\} \mathrm{d}\mathbf{x}. \quad (40)$$

We now assume that (remember that Z_q is just a number and does not depend on $\mathbf{x}$)

$$\int \left(L_{\mathrm{R}} f_{\mathrm{ME}} \right) Z_q \mathrm{d}\mathbf{x} = \int \left(L_{\mathrm{R}}^{\dagger} Z_q \right) f_{\mathrm{ME}} \mathrm{d}\mathbf{x} = 0, \quad (41)$$

and

$$\int \left(L_{\mathrm{D}} f_{\mathrm{ME}} \right) Z_q \mathrm{d}\mathbf{x} = \int \left(L_{\mathrm{D}}^{\dagger} Z_q \right) f_{\mathrm{ME}} \mathrm{d}\mathbf{x} = 0. \quad (42)$$

These last two equations involve an integration by parts procedure. We assume that f_{ME} verifies appropriate boundary conditions (essentially, it goes to zero fast enough with $|\mathbf{x}| \to 0$) in order for the "integrated part" being zero. Then we obtain:

$$\begin{aligned} \frac{\mathrm{d}S_q\left[f_{\mathrm{ME}}\right]}{\mathrm{d}t} &= \frac{q}{1-q} \int \left\{ L_{\mathrm{R}} f_{\mathrm{ME}} + L_{\mathrm{D}} \left(f_{\mathrm{ME}}^{2-q} \right) \right\} f_{\mathrm{ME}}^{q-1} \mathrm{d}\mathbf{x} \\ &= D_q\left[f_{\mathrm{ME}}\right], \end{aligned} \quad (43)$$

and have therewith proved that

$$\frac{\mathrm{d}S_q[f_{\mathrm{ME}}]}{\mathrm{d}t} = D_q\left[f_{\mathrm{ME}}\right]. \quad (44)$$

We can conclude that the functional relation giving the time derivative of the Tsallis entropy S_q in terms of the approximate MaxEnt ansatz f_{ME} is the same as the one verified in the case of the unknown *exact solutions*. This important property is verified in general, for any (exact) solution of the nonlinear Fokker-Planck equation and regardless of the particular set of relevant mean values $\langle A_i \rangle_q$ employed in order to build up the corresponding Tsallis maximum entropy approximation. The present derivation explicitly makes use of the q-MaxEnt form of the (approximate) q-MaxEnt solution. However, it is possible that a similar property holds within a more general context involving other kinds of evolution equations endowed with an associated "natural" entropic measure. This possibility is currently under consideration.

4.3 Hamiltonian structure

We shall now obtain the equations of motion for the Lagrange multipliers λ_i, and study how are they related to the equations of motion of the concomitant mean values $\langle A_i \rangle_q$. Making use of Jaynes' thermodynamic relations, we have

$$\begin{aligned} \frac{\mathrm{d}\lambda_i}{\mathrm{d}t} &= \sum_{j=1}^{M} \frac{\partial \lambda_i}{\partial \langle A_j \rangle_q} \frac{\mathrm{d}\langle A_j \rangle_q}{\mathrm{d}t} \\ &= \sum_{j=1}^{M} \frac{\partial^2 S_q}{\partial \langle A_i \rangle_q \partial \langle A_j \rangle_q} \frac{\mathrm{d}\langle A_j \rangle_q}{\mathrm{d}t} \\ &= \sum_{j=1}^{M} \frac{\partial \lambda_j}{\partial \langle A_i \rangle_q} \frac{\mathrm{d}\langle A_j \rangle_q}{\mathrm{d}t}, \end{aligned} \quad (45)$$

which can be recast as

$$\begin{aligned} \frac{\mathrm{d}\lambda_i}{\mathrm{d}t} &= \frac{\partial}{\partial \langle A_i \rangle_q} \left[\sum_{j=1}^{M} \lambda_j \frac{\mathrm{d}\langle A_j \rangle_q}{\mathrm{d}t} \right] \\ &\quad - \sum_{j=1}^{M} \lambda_j \frac{\partial}{\partial \langle A_i \rangle_q} \left(\frac{\mathrm{d}\langle A_j \rangle_q}{\mathrm{d}t} \right). \end{aligned} \quad (46)$$

Making use now of the equations of motion (28) for the relevant mean values, and of the expression (37) for the time derivative of the entropy, we finally obtain

$$\begin{aligned} \frac{\mathrm{d}\lambda_i}{\mathrm{d}t} &= \frac{\partial}{\partial \langle A_i \rangle_q} \left[\frac{\mathrm{d}Sq[f_{\mathrm{ME}}]}{\mathrm{d}t} \right] \\ &\quad - q \sum_{j=1}^{M} \lambda_j \frac{\partial}{\partial \langle A_i \rangle_q} \int \left\{ L_{\mathrm{R}} f_{\mathrm{ME}} + L_{\mathrm{D}} \left(f_{\mathrm{ME}}^{2-q} \right) \right\} f_{\mathrm{ME}}^{q-1} A_j \mathrm{d}\mathbf{x}. \end{aligned} \quad (47)$$

Introducing now the Hamiltonian

$$H(\langle A_1 \rangle_q, \ldots, \langle A_M \rangle_q, \lambda_1, \ldots \lambda_M) = D_q[f_{\mathrm{ME}}] - q \sum_{j=1}^{M} \lambda_j \int \left\{ L_{\mathrm{R}} f_{\mathrm{ME}} + L_{\mathrm{D}} \left(f_{\mathrm{ME}}^{2-q} \right) \right\} f_{\mathrm{ME}}^{q-1} A_j \mathrm{d}\mathbf{x}, \quad (48)$$

the equations of motion for the relevant mean values and their associated Lagrange multipliers can be put in a Hamiltonian way,

$$\frac{\mathrm{d}\langle A_i \rangle_q}{\mathrm{d}t} = -\frac{\partial H}{\partial \lambda_i}, \quad (49)$$

and

$$\frac{\mathrm{d}\lambda_i}{\mathrm{d}t} = \frac{\partial H}{\partial \langle A_i \rangle_q} \cdot \tag{50}$$

In the expression (48) defining our Hamiltonian function, the MaxEnt distribution f_{ME} should be regarded as parametrized by the set of M relevant mean values $\langle A_i \rangle_q$. That is to say, the specific values adopted by the M quantities $\langle A_i \rangle_q$ determine, via the MaxEnt recipe, a particular distribution f_{ME}. In this way, the functionals of f_{ME} appearing in (48) are functions of the M quantities $\langle A_i \rangle_q$. Consequently, the only dependence of the Hamiltonian on the Lagrange multipliers is the one explicitly shown in equation (48). It is important to realize that ours is not a time dependent Hamiltonian, since its functional dependence on the relevant mean values and their associated Lagrange multipliers is dictated by the MaxEnt procedure and does not depend on time.

Note also that although all the solutions of our time dependent MaxEnt scheme evolve according to the Hamiltonian equations (49, 50), not all the orbits associated with the Hamiltonian (48) constitute realizations of our MaxEnt approach. The orbits that are relevant to our problem are those whose initial conditions verify

$$\langle A_i \rangle_q = \frac{1}{Z_q^q} \int \mathrm{d}\mathbf{x}\, A_i(\mathbf{x}) \left[1 - (1-q) \sum_{i=1}^{M} \lambda_i A_i(\mathbf{x}) \right]^{q/(1-q)}, \quad i = 1, \ldots, M. \tag{51}$$

These M equations determine an M-dimensional submanifold of the full $2M$ phase space $(\langle A_1 \rangle_q, \ldots, \langle A_M \rangle_q, \lambda_1, \ldots, \lambda_M)$. This submanifold constitutes an invariant set of our Hamiltonian dynamical system. That is to say, any orbit with initial conditions belonging to the set stays within it forever.

Summing up, we conclude that within the present time dependent thermostatistical context the relevant mean values and their concomitant Lagrange multipliers are conjugate variables not only in the thermodynamical sense, but also in the Hamiltonian phase space sense.

4.4 Variational treatment

We shall now introduce an appropriate variational principle for the nonlinear Fokker-Planck equation. In analogy to the treatment of the linear Fokker-Planck equation using the standard Boltzmann-Gibbs entropy [55], we propose the q-generalized action

$$K = q \int_{t_1}^{t_2} \mathrm{d}t \int \mathrm{d}\mathbf{x} f^{q-1} g(\mathbf{x},t) \left\{ \frac{\partial f}{\partial t} - L_{\mathrm{R}} f - L_{\mathrm{D}} \left(f^{2-q}\right) \right\} + \int_{t_1}^{t_2} D_q\,[f]\, \mathrm{d}t - \int f^q(\mathbf{x},t_2) g(\mathbf{x},t_2) \mathrm{d}\mathbf{x}, \tag{52}$$

where an auxiliary quantity $g(\mathbf{x},t)$ has been introduced. The variational principle should provide us with equations of motion for both $f(\mathbf{x},t)$ and $g(\mathbf{x},t)$. The need to introduce auxiliary variables in order to formulate an action principle for systems showing diffusive or dissipative behaviour occurs in many problems of mathematical physics [60]. Usually, there exists a mathematical transformation relating the auxiliary variable with the original one [60]. We will see that this is indeed the case with the present action principle.

We shall use the "mixed" boundary conditions

$$\begin{aligned} f(\mathbf{x},t_1) &= f(\mathbf{x})_{\mathrm{in}} \\ g(\mathbf{x},t_2) &= g(\mathbf{x})_{\mathrm{out}}. \end{aligned} \tag{53}$$

The action principle

$$(\delta K)_{f,g} = 0 \tag{54}$$

leads to the partial differential equation for $f(\mathbf{x},t)$ and the auxiliary variable $g(\mathbf{x},t)$.

For arbitrary δg,

$$(\delta K)_g g = 0 \Rightarrow f^{q-1} \left\{ \frac{\partial f}{\partial t} - L_{\mathrm{R}} f - L_{\mathrm{D}} \left(f^{2-q}\right) \right\} = 0, \tag{55}$$

so that

$$\frac{\partial f}{\partial t} - L_{\mathrm{R}} f - L_{\mathrm{D}} \left(f^{2-q}\right) = 0 \tag{56}$$

is the non-linear Fokker-Planck equation.

For arbitrary δf,

$$\begin{aligned} (\delta K)_f = q \int_{t_1}^{t_2} \mathrm{d}t \int \mathrm{d}\mathbf{x} g(\mathbf{x},t) \delta(f^{q-1}) & \\ \times \left\{ \frac{\partial f}{\partial t} - L_{\mathrm{R}} f - L_{\mathrm{D}} \left(f^{2-q}\right) \right\} & \\ + q \int_{t_1}^{t_2} \mathrm{d}t \int \mathrm{d}\mathbf{x} g(\mathbf{x},t) f^{q-1} \delta & \\ \times \left\{ \frac{\partial f}{\partial t} - L_{\mathrm{R}} f - L_{\mathrm{D}}(f^{2-q}) \right\} & \\ + \int_{t_1}^{t_2} \mathrm{d}t\ \delta D_q[f] = 0. & \end{aligned} \tag{57}$$

The first term of right hand side, because of (55), makes no contribution. Then,

$$\begin{aligned} (\delta K)_f = 0 \Rightarrow \frac{\partial \left(g f^{q-1}\right)}{\partial t} & \\ + L_{\mathrm{R}}^{\dagger}(g f^{q-1}) + (2-q) f^{1-q} L_{\mathrm{D}}^{\dagger}(g f^{q-1}) & \\ - \frac{1}{1-q} \left\{ L_{\mathrm{R}}^{\dagger}(f^{q-1}) + (2-q) f^{1-q} L_{\mathrm{D}}^{\dagger}(f^{q-1}) \right\} & \\ + f^{q-2} \left\{ L_{\mathrm{R}}(f) + L_{\mathrm{D}}(f^{2-q}) \right\} = 0. & \end{aligned} \tag{58}$$

Summing up, both $f(x,t)$ and $g(x,t)$ verify the coupled equations

$$\frac{\partial f}{\partial t} - L_{\mathrm{R}} f - L_{\mathrm{D}} \left(f^{2-q}\right) = 0, \tag{59}$$

and

$$\frac{\partial \left(g f^{q-1}\right)}{\partial t} + L_{\mathrm{R}}^{\dagger}(g f^{q-1}) + (2-q) f^{1-q} L_{\mathrm{D}}^{\dagger}(g f^{q-1})$$
$$-\frac{1}{1-q}\left\{L_{\mathrm{R}}^{\dagger}(f^{q-1}) + (2-q) f^{1-q} L_{\mathrm{D}}^{\dagger}(f^{q-1})\right\}$$
$$+f^{q-2}\left\{L_{\mathrm{R}}(f) + L_{\mathrm{D}}(f^{2-q})\right\} = 0. \tag{60}$$

These two evolution equations are closely related. Given a time dependent solution $f(\mathbf{x},t)$ of the first equation (59), it is easy to verify that

$$g(\mathbf{x},t) = \frac{1 - f^{1-q}}{1-q}, \tag{61}$$

provides a solution to equation (60). We can rewrite this last relation in terms of the generalized functions that have been recently introduced inspired by Tsallis Thermostatistics [61],

$$g = -\ln_q f. \tag{62}$$

This equation shows that there is a simple mathematical connection between f and g related to the Tsallis q-formalism.

Finally, we conclude this section by pointing out again that all of the above results in the q-generalized maximum entropy derivation can alternatively be obtained using the normalized q-mean values $\langle\langle A_i \rangle\rangle_q \equiv \langle A_i \rangle_q / \int f^q \mathrm{d}\mathbf{x}$ instead of the q-averages $\langle A_i \rangle_q$ defined in equation (27). Equivalent results are also found by using standard averages $\langle A_i \rangle = \int f A_i(\mathbf{x}) \mathrm{d}\mathbf{x}$. The only difference (apart from philosophical ones) is manifested in that $q-1$ is replaced by $1-q$.

5 Examples

5.1 Exact time-dependent solutions

In the following we shall illustrate the above ideas by seeking maximum entropy time-dependent solutions to the one-dimensional nonlinear Fokker-Planck equation with x-dependent diffusion. There has already been some recent work in studying solutions of the nonlinear Fokker-Planck equation with *constant* diffusion coefficient Q:

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x}(K(x) f) + Q \frac{\partial^2 f^{\alpha}}{\partial x^2}. \tag{63}$$

Exact solutions have been proposed in [27,42]. The stationary solutions are of the form

$$f(x) = \Gamma[1 - \beta(1-q) V(x)]^{1/(1-q)}, \tag{64}$$

with $q = 2 - \alpha$, $V(x) = -\int K(x)\mathrm{d}x$, $\beta = \frac{\Gamma^{q-1}}{\alpha Q}$ and $\Gamma = 1/Z_q$ where Z_q is the generalized partition function. To be physically reasonable, q may take on any real number $q < 3$, above which f has normalization problems [42]. There is a singularity in the nonlinear Fokker-Planck equation at $q = 2$ but physically relevant solutions can still be found [44]. The solutions (21), referred to as Tsallis distributions, are such that they maximize the nonextensive generalized Tsallis entropy equation (1). The time-dependent solutions are also of the same general form, but now one has time-dependent coefficients such as $\beta(t)$ and $\Gamma(t)$. For all these solutions there is a cut-off if the argument in square brackets becomes negative. In this case, $f(x,t) = 0$.

Let us now seek solutions to the more general nonlinear Fokker-Planck equation of the form (19), namely

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x}(K(x) f) + \frac{\partial}{\partial x}\left(D(x)\frac{\partial f^{\alpha}}{\partial x}\right). \tag{65}$$

The stationary solutions must satisfy

$$\frac{\partial}{\partial x}\left[-K(x) f + D(x)\frac{\partial f^{\alpha}}{\partial x}\right] = 0, \tag{66}$$

so that the expression in square brackets must be a constant, which can be made to vanish by assuming appropriate boundary conditions. This yields

$$\alpha f^{\alpha-2}\mathrm{d}f = \frac{K(x)}{D(x)}\mathrm{d}x, \tag{67}$$

which can be solved to give the expression already shown in equation (21), namely

$$f(x) = \Gamma[1 - \beta(1-q) V_{\mathrm{eff}}(x)]^{1/(1-q)}, \tag{68}$$

with the effective potential

$$V_{\mathrm{eff}} = -\int \frac{K(x)}{D(x)}\mathrm{d}x, \tag{69}$$

and the constant $\beta = \Gamma^{q-1}/Q\alpha$. This shows that for the nonlinear Fokker-Planck equation of type (65), the stationary solutions are of the same Tsallis form as the case $D(x) = Q$.

Now we begin our quest for some time-dependent solutions to equation (65). Let us consider drift and diffusion coefficients of the form

$$K(x) = -c_{\mathrm{K}} x^{\gamma} \tag{70}$$

and

$$D(x) = c_{\mathrm{D}} x^{r}, \tag{71}$$

where γ and r are real numbers. This corresponds to the physical situation of *multiplicative* noise, where x is usually limited to $x > 0$. Note that one may include the $x < 0$ regime also by redefining $K = |x|^{\gamma-1}x$ and $D = |x|^r$. For the standard linear Fokker-Planck equation, similiar problems have been studied for example by [62,63]. With our choice of K and D, we have the relation $s \equiv \gamma - r + 1$, where s corresponds to the power of x in the effective potential

$$V_{\rm eff}(x) = \frac{c_{\rm K}}{c_{\rm D}} \int \frac{x^\gamma}{x^r} {\rm d}x \propto x^s. \tag{72}$$

Our time-dependent maximum entropy ansatz is chosen as

$$f(x,t) = \Gamma(t)[1-(1-q)\beta(t)x^s]^{1/(1-q)}. \tag{73}$$

(Note that the most general maximum entropy ansatz would contain instead of βx^s a sum of the type $\sum \lambda_\nu(t)x^\nu$ where each power of x is associated with a Lagrange multiplier λ. For the sake of simplicity we choose not to discuss this case here.)

Let us now insert equation (73) into equation (65). We obtain:

$$\frac{\partial f}{\partial t} = \dot{\Gamma} A^{1/1-q} - \Gamma\dot{\beta} x^s A^{q/1-q} \tag{74}$$

$$\frac{\partial}{\partial x}(Kf) = -\gamma_{\rm K} x^{\gamma-1}\Gamma A^{1/1-q} + c_{\rm K}\beta\Gamma s x^{\gamma+s-1}A^{q/1-q} \tag{75}$$

$$\frac{\partial}{\partial x}\left(D\frac{\partial}{\partial x}f^{2-q}\right) = c_{\rm D}(q-2)\beta s(s+r-1)\Gamma^{2-q}x^{s+r-2}A^{1/1-q} - c_{\rm D}(q-2)\beta^2 s^2\Gamma^{2-q}x^{2s+r-2}A^{q/1-q}, \tag{76}$$

where $A = 1-(1-q)\beta x^s$, and we have substituted $\alpha = 2-q$. The question now is if we can find values of γ and r so as to obtain a set of unique solutions to the equations (74–75). Our study shows that this is indeed the case if $\gamma = 1$ and $r+s = 2$, which results in terms of the type $A^{1/(1-q)}$, $A^{q/(1-q)}$ and $x^s A^{q/(1-q)}$ in the expressions (74–75). Comparing coefficients then gives the following relations:

$$\frac{{\rm d}\Gamma}{{\rm d}t} = c_{\rm K}\Gamma - s\beta c_{\rm D}(2-q)\Gamma^{2-q} \tag{77}$$

$$\frac{{\rm d}\beta}{{\rm d}t} = c_{\rm K}s\beta - c_{\rm D}(2-q)\Gamma^{1-q}s^2\beta^2, \tag{78}$$

which results in

$$\frac{{\rm d}\beta}{{\rm d}t} = s\frac{\beta}{\Gamma}\frac{{\rm d}\Gamma}{{\rm d}t}, \tag{79}$$

so that, after integration, one finds

$$\beta(t) = \beta(t_0)\left(\frac{\Gamma(t)}{\Gamma(t_0)}\right)^s, \tag{80}$$

where t_0 is an initial time which we set to $t_0 = 0$ in the following. Expression (80) can be inserted into equation (77) to obtain the following differential equation for Γ

$$\frac{{\rm d}\Gamma}{{\rm d}t} = c_{\rm K}\Gamma - s\frac{\beta(t_0)}{\Gamma(t_0)^s}c_{\rm D}(2-q)\Gamma^{2+s-q}, \tag{81}$$

which can be solved to give

$$\Gamma(t) = \Gamma(t_0)\left[(1-\delta){\rm e}^{-t/\tau} + \delta\right]^{-1/(1+s-q)}, \tag{82}$$

with

$$\delta = c_{\rm D}(2-q)s\beta(t_0)\Gamma(t_0)^{1-q}/c_{\rm K} \tag{83}$$

and

$$1/\tau = c_{\rm K}(1+s-q). \tag{84}$$

(Note that the parameters in Eq. (83) must be chosen so that $\delta < 1$ to obtain physically relevant solutions). It follows then from equation (80) that

$$\beta(t) = \beta(t_0)\left[(1-\delta){\rm e}^{-t/\tau} + \delta\right]^{-s/(1+s-q)}. \tag{85}$$

The time-dependent solution to the problem of nonlinear diffusion with linear drift $K(x) = -c_{\rm K}x$ (remember $\gamma = 1$) and nonlinear diffusion $D(x) = c_{\rm D}x^r$ is thereby given by the ansatz in equation (73) (with $s = 2-r$) together with the evolution equations (82, 85). In Figures 1 and 2 we show some examples of time-dependent solutions for different q and s.

Let us now study the important case of free diffusion, which occurs when $c_{\rm K} = 0$. In this case we obtain from equation (77) the solution

$$\Gamma(t) = \left[\Gamma_0^{q-s-1} + c_{\rm D}\frac{\beta_0}{\Gamma_0^s}s(2-q)(1+s-q)t\right]^{-1/(1+s-q)}. \tag{86}$$

The corresponding expression for $\beta(t)$ can then be obtained straightforwardly from equation (80). We see that in both the cases of free and confined diffusion ($c_{\rm K} \neq 0$), the two effects stemming from s and q combine to determine the temporal behaviour of Γ and β: the $s = 2-r$-term is determined by the power r of the x-dependent nonlinear diffusion coefficient, whereas the $q = 2-\alpha$ term is related to the power α of the nonlinearity in f. These two effects will ultimately determine the rate of the anomalous diffusion of the process. This can be seen more explicitly by calculating $\langle x^2\rangle(t)$ in the limit $t \to \infty$, for the case of free diffusion. We obtain

$$\langle x^2\rangle = \int x^2 f(x,t){\rm d}x \tag{87}$$

$$= \Gamma\int x^2[1-(1-q)\beta x^s]^{1/(1-q)}{\rm d}x, \tag{88}$$

which can be greatly simplified by introducing the new integration variable $u = [(1-q)\beta]^{1/s}x$ if $1-q > 0$. (If $1-q < 0$ introduce instead $u = [-(1-q)\beta]^{1/s}x$ and proceed along the same lines.) Consequently we find

$$\langle x^2\rangle = \Gamma[(1-q)\beta]^{-3/s}\int u^2(1-u^s)^{1/(1-q)}{\rm d}u. \tag{89}$$

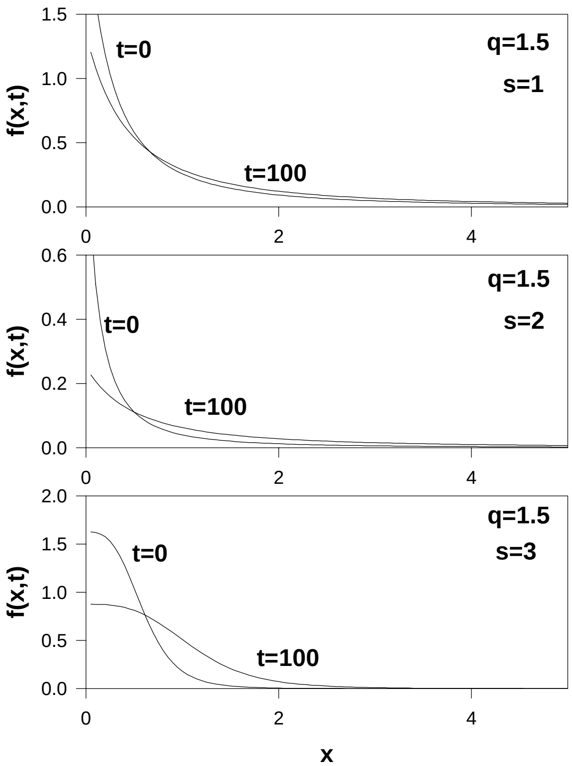

Fig. 1. Solutions $f(x,t)$ to the nonlinear Fokker-Planck equation with linear drift and multiplicative nonlinear power-law diffusion of type x^r, with $r = s - 2$ (where s is the power of x in the *effective potential* $V_{\text{eff}} = x^s$). Here we see $f(x,t)$ for $q = 1.5$ and three different values of s: $s = 2$ corresponds to the case of constant diffusion, while $s = 1$ yields a diffusion function $1/x$ and $s = 3$ represents multiplicative noise of type x. Note how both the tails (for large x) and the curvature (for small x) change with s. Stationary solutions correspond to those at long times (*e.g.* $t = 100$).

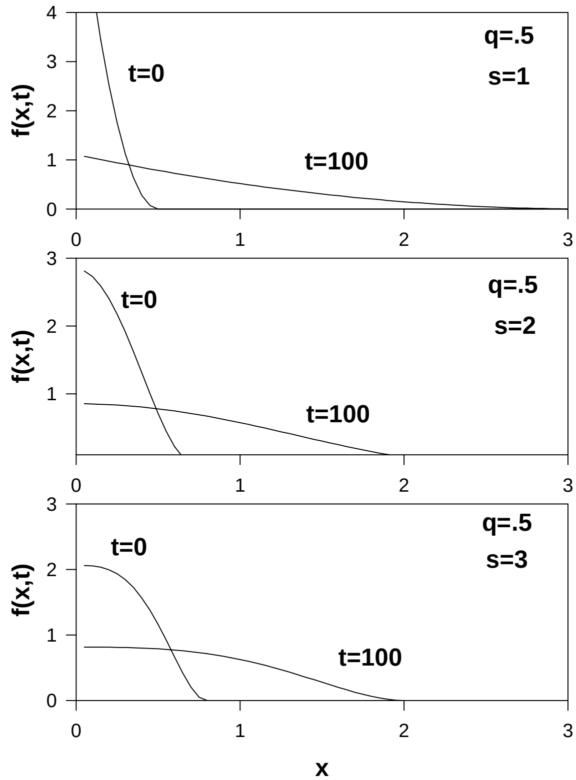

Fig. 2. Here we show $f(x,t)$ corresponding to the nonlinear Fokker-Planck equation with linear drift and three types of multiplicative noise x^r ($r = s-2$, with effective potential $V_{\text{eff}} = x^s$) at $q = 0.5$: $s = 1$ corresponds to $1/x$ noise, $s = 2$ is constant noise and $s = 3$ is linear multiplicative noise x. Notice how the distribution changes shape as s increases. Stationary solutions correspond to the long-time solutions shown here ($t = 100$).

But the last integral now depends only on u and is therefore irrelevant in our discussion on temporal behaviour which is contained in $\Gamma(t)$ and $\beta(t)$ as given by equation (86) together with equation (80). Asymptotically then, our calculation yields

$$\langle x^2 \rangle(t) \propto t^{2/(1+s-q)} = t^{2/(1+\alpha-r)}. \tag{90}$$

This implies that we obtain *normal* diffusion for $s = q+1$ (*i.e.*, $\alpha - r = 1$), *sub*-diffusion for $s > q+1$ (*i.e.*, $r > \alpha - 1$) and *super*-diffusion for $s < q+1$ (*i.e.*, $r < \alpha - 1$). For the special choice of $s = 2$ which corresponds to constant diffusion, we see that this result coincides with the previous results obtained in [43,44].

Note that in the above we calculated $\langle x^2 \rangle$ using *standard* averages (taken over f) and not generalized q-averages (taken over f^q). We proceed in such a fashion because the definition of the anomalous diffusion exponent is based on the standard average, used when the only available information about the system under study consist of certain observations, such as a time-series of the state-variable x. However, it is interesting to take a look at the generalized $\langle x^2 \rangle_q(t)$ averages as well. We have

$$\langle x^2 \rangle_q = \int x^2 f^q(x,t)\mathrm{d}x \tag{91}$$

$$= \Gamma^q \int x^2 [1 - (1-q)\beta x^s]^{q/(1-q)} \mathrm{d}x, \tag{92}$$

which yields

$$\langle x^2 \rangle_q(t) \propto t^{(3-q)/(1+s-q)} = t^{(1+\alpha)/(1+\alpha-r)}. \tag{93}$$

We see that for constant diffusion ($s = 2$ *i.e.*, $r = 0$) the diffusion exponent becomes *equal to 1 for all values of* q (and therewith α). This implies that the system diffuses normally with respect to the subspace of phase-space captured by the f^q representation, where rare and common events are weighted differently. It also implies a q-invariance in this representation which exists beyond the $s = 2$ (*i.e.*, $r = 0$) case in the sense that we find – independently of q – normal diffusion for $s = 2$ (*i.e.*, $r = 0$), sub-diffusion for $s > 2$ (*i.e.*, $r < 0$) and super-diffusion for $s < 2$ (*i.e.*, $r > 0$). The actual value of the diffusion exponent does however still depend on q for general $s \neq 2$. Some plots of $\langle x^2 \rangle(t)$ and $\langle x^2 \rangle_q(t)$ for different q and s are shown in Figures 3 and 4.

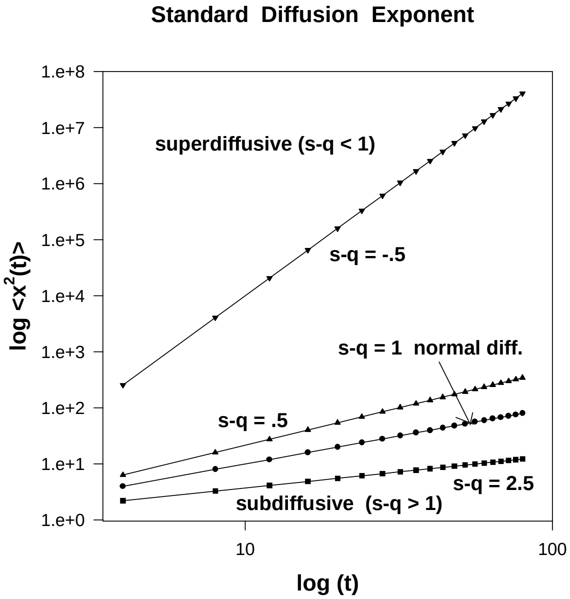

Fig. 3. The diffusion exponent γ is defined as $\langle x^2\rangle(t) \propto t^{\gamma}$. γ corresponds to the slope in this (asymptotic) log-log plot for free diffusion, and we clearly see that it is determined by both q and s. In particular we have normal diffusion ($\gamma = 1$) for $s-q = 1$, subdiffusion ($\gamma < 1$) for $s-q > 1$ and superdiffusion ($\gamma > 1$) for $s-q < 1$. Some lines for particular values of q and s are shown. s corresponds to the power of the multiplicative noise which is given by x^{s-2}.

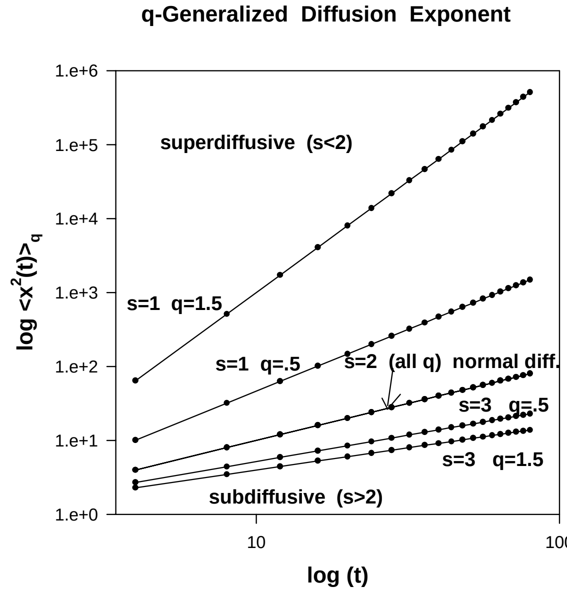

Fig. 4. Within the generalized Tsallis thermostatistics, it is also interesting to look the q-generalized time dependent moment $\langle x^2\rangle_q(t) \equiv \int x^2 f^q \mathrm{d}x \propto t^{\gamma_q}$. We can calculate the corresponding q-generalized diffusion exponent γ_q as the slope in this log-log plot, calculated for free diffusion with multiplicative noise x^{s-2}. We find a q-invariant behaviour in the following sense: Normal diffusion is always obtained for $s = 2$ (constant diffusion) independent of q. Similarly, there is subdiffusion ($\gamma_q < 1$) for $s > 2$ and superdiffusion ($\gamma_q > 1$) for $s < 2$. Only the actual value of γ_q depends on q, but not the type of diffusion. This last q-dependent effect is only seen for multiplicative noise ($s \neq 2$).

Figuratively speaking, these results tell us that, in the case of constant diffusion, the underlying dynamics of the system are confined to the f^q space, within which everything acts as a normal, linear diffusive system. If we view the system through f^q weighted "glasses", then we also see it as a normally diffusing system, q-invariant and completely analogous to a standard $q = 1$ system. However, if we have no knowledge neither of the underlying dynamics nor of which f^q glasses to choose (as is most often the case!) we simply view the system through ordinary homogeneously weighted f glasses. In doing so we perceive all kinds of q dependent anomalies. If the diffusion is state-dependent, the same behaviour still holds, the only difference being that the actual value of the diffusion exponent is not q-invariant in the f^q-case. It is in itself interesting to note that the mere inclusion of a multiplicative noise term captures this q-effect, otherwise unseen in the f^q representation.

Finally, it is instructive to compare the behaviour of q-mean values within the nonlinear Fokker-Planck equation with their behaviour in the case of the Liouville equation associated with a Hamiltonian $H(q_i, p_i)$ (the Hamiltonian systems we are considering here should not be confused with the Hamiltonian structure discussed in Section 4. That structure was associated with the "macroscopic" evolution of the parameters characterizing the MaxEnt solutions of the nonlinear Fokker-Planck equation. However, solutions of the Fokker-Planck equation do not describe an ensemble of independent systems evolving according to an underlying Hamiltonian dynamics, as is the case in the Hamilton-Liouville context). It is easy to verify that for any time dependent solution $f(q_i, p_i, t)$ of Liouville's equation, $f^q(q_i, p_i, t)$ is also a solution. Therefore, the generalized q-mean values $\langle B\rangle_q = \int f^q(q_i, p_i, t) B(q_i, p_i) \mathrm{d}\Omega$ behave essentially in the same way as the standard linear mean values. A similar situation happens in the case of the von Neumann equation associated with quantum Hamiltonian systems, which yields the q-generalization of Eherenfest's Theorem [13]. Hence, within this Hamilton-Liouville (or von Neumann, in the quantum case) context we are allowed to say that the macroscopic dynamics is completely "q-invariant". For example, if we have a free particle, both $\langle x^2\rangle$ and $\langle x^2\rangle_q$ grow as t^2. The reason for this full "q-invariance" boils down to two simple properties of the (Hamiltonian) Liouville equation. First of all, it is a *linear* evolution equation. Secondly, it is associated with a dynamical system exhibiting a *divergenceless* phase space flow [54]. In order to clarify this last assertion, let's consider the more general case of divergenceless dynamical system [54]

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{v}(\mathbf{x}), \quad \mathbf{x}, \mathbf{v} \in \mathrm{R}^N, \tag{94}$$

where we assume a divergenceless flow in phase space

$$\nabla \cdot \mathbf{v} = 0. \tag{95}$$

Hamiltonian systems constitute particular instances of this family of dynamical systems [54]. The concomitant Liouville equation reads

$$\frac{\partial f}{\partial t} + \nabla \cdot (\mathbf{v} f) = \frac{\partial f}{\partial t} + \mathbf{v} \cdot (\nabla f) = 0, \quad (96)$$

where the divergenceless condition was used. Now, the last equality can be recast under the guise

$$\left(\frac{\mathrm{d}f(\mathbf{x},t)}{\mathrm{d}t}\right)_{\text{Orbit}} = 0, \quad (97)$$

where the time derivative in the last equation is evaluated along a particular orbit of the dynamical system (94). It is now easy to see that given a solution $f(\mathbf{x},t)$ of (97), any power f^q will satisfy that equation too, and full q-invariance will ensue. On the contrary, this complete "q-invariance" will be lost if we deal with *nonlinear* evolution equations, or even with linear Liouville equations arising from a phase space flow with nonvanishing divergence.

5.2 Another possible class of solutions

We now conclude with a brief discussion of a slightly different class of solutions, mainly in order to make an exact connection with the previously studied solutions to the one-dimensional nonlinear Fokker-Planck equation with constant diffusion. These contain earlier results on the time-dependent solutions to the nonlinear Fokker-Planck equation as a particular case. The ansatz we look at now is of the form

$$f(x,t) = \Gamma(t)[1-(1-q)\beta(t)(x-x_0(t))^s]^{1/(1-q)}, \quad (98)$$

which is the same as the maximum entropy ansatz used above except for the $x_0(t)$ term. Studying solutions of this type is important because it is a possible generalization of the Gaussian case that incorporates in a natural way many powers of x as constraints. Note that this ansatz is also a maximum entropy ansatz, just rewritten so that in certain instances the Lagrange multipliers in the standard formalism are split into products of $\beta(t)$ and powers of $x_0(t)$.

By inserting equation (98) into the nonlinear Fokker-Planck equation (65) one can perform an analysis completely analogous to the example discussed above. Our study shows that there are only two sets of possible parameter choices which yield solutions. One set is given by $r = 0, s = 2$ and $\gamma = 1$. This however corresponds to constant diffusion and results just in the solutions found by Plastino and Plastino [27] and Tsallis and Bukman [42]. The second set of parameters which admit a solution is given by $s = r = \gamma = 1$. But this solution can be rewritten, without loss of generality, such that $x_0 = 0$. It then takes on the same form as the $s = 1$ solution found above using the ansatz (73).

6 Conclusions

The main point of this paper was to study the nonlinear Fokker-Planck equation with general x-dependent diffusion, thus generalizing the case of constant diffusion which has been discussed previously [27,42]. An approximate maximum entropy approach was developed, based on the nonextensive Tsallis entropy, together with a variational action principle. It was found that if the nonlinear Fokker-Planck equation is of a Stratonovich-like type (as opposed to the Ito form) then the general stationary solutions are of the type that maximize the Tsallis entropy, comparable to the way in which Gaussian distributions which maximize the standard Shannon entropy are stationary solutions to the linear Fokker-Planck equation. Our analysis supports the idea that the Stratonovich form is the more physically relevant one, a conclusion also found in the case of the linear Fokker-Planck equation due to the fact that the Stratonovich approach includes a more realistic treatment of the noise [39,40]. Also true for the nonlinear Stratonovich form is that the maximum entropy solutions preserve the functional relation between the time derivative of the entropy and the time dependent solution. This fact is quite remarkable. In addition, exact time-dependent solutions were found for some physically interesting cases, namely linear drift and power-law nonlinear multiplicative noise. We found that both the degree of nonlinearity in the Fokker-Planck equation itself together with the power of the multiplicative noise combine to ultimately determine the rate of anomalous diffusion of the process. However, we also studied a q-generalized diffusion exponent, based on f^q averages rather than the standard f average, and found that important properties of the diffusion are q-invariant in that representation: Whether normal-, super-, or subdiffusive depends only on the power of the multiplicative noise. Further investigation of a q-generalized diffusion exponent based on the recently introduced *normalized* q-averages [59] is currently underway.

The authors thank C. Tsallis for helpful and inspiring comments. A.R.P. and A.P. acknowledge support from CONICET (Argentine Agency). F.P. acknowledges support from the National University La Plata (UNLP). L.B. acknowledges support from the U.S. National Science Foundation. This material is based upon work supported by the National Science Foundation under Grant No. INT-9703649. Partial support from PRONEX (Brazilian agency) is acknowledged as well.

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